Application of Sumudu Transform in Advanced Engineering Mathematics Problems

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ABSTRACT:

In this paper we will discuss mechanics and electrical circuit problems in term of a differential equation in the field of engineering. The solutions of these differential equations are obtained by Sumudu transform. Sumudu transform is a very useful and powerful mathematical tool for solving many advanced problems of engineering and applied sciences. The Sumudu Transform was applied to solve differential equations. These equations are concerned with a damping mechanical force system and an inductive capacitive electric circuit.

KEYWORD: Differential equation, Sumudu Transform, Inverse Sumudu Transform, Kirchhoff's law.

INTRODUCTION:

The mathematical techniques used differential equations, these techniques successfully solve many physical problems associated with the behavior of the physical system. These techniques are widely used in classical mechanical and quantum mechanical problems. The solution of these equations needs knowing initial or boundary conditions or physical and mathematical constraints imposed by the physical system [13-14]. The importance of mathematical techniques encourages many researches in mathematic to propose new techniques for solving physical problems [12].

In the literature there are several works on the application of integral transform such as Laplace transform, Mohgoub Transform, Aboodh Transform, Kamal Transform, Elzaki Transform to name a few, but very little on the power series transformation such as Sumudu Transform because it is little known and not widely used yet. The Sumudu Transform was proposed originally by Watugala (1993) to solve differential equation and control engineering problems [6].

The Weerakoon [5] paper, showing Sumudu Transform applications to partial differential equations, immediately followed Watugala's [1] seminal work. Watugala's [2] work showed that the Sumudu Transform can be effectively used to solve ordinary differential equations and engineering control problems.

The purpose of this study is to show the applicability of this transform and it's efficiency in solving the linear differential equations.

In the paper given few examples to understand the new method, Application of this new transform can used to solve mathematical mechanics and electrical circuit problems of linear differential equation with constant coefficient.

A new integral transform called the Sumudu transform is introduced in 1993 by G.K. watugala. This transform is defined over the set of function [1-3].

$$A = \left\{ f(t): \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{t}{\tau}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

For a given function in the set A, the constant M must be finite, while τ_1 and τ_2 need not simultaneously exit, and each may be infinite. The Sumudu transform of time function $f(t)$ denoted by $S[f(t): u]$ or simply $G(u)$ then above equation can be reduced to following form.

$$S\{f(t)\} = G(u) = \frac{1}{u} \int_0^\infty e^{\frac{-t}{u}} f(t) dt$$

where u is a parameter and it may be real or complex and it is independent of t [2-4].

The inverse Sumudu transform of function $G(u)$ is denoted by symbol $S^{-1}[G(u)] = f(t)$ and defined with Bromwich contour integral [2].

$$S^{-1}[G(u)] = f(t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{st} G(u) du$$

Sumudu transform of the function derivatives [10-11].

$$G_n = \frac{1}{u^n} G(u) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} = \frac{1}{u^n} \left[G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0) \right]$$

$$G_1(u) = S[f'(t)] = \frac{G(u)}{u} - \frac{f(0)}{u}$$

$$G_2(u) = S[f''(t)] = \frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u}$$

$$G_n(u) = S\left[f^{(n)}(t)\right] = \frac{G(u)}{u^n} - \frac{f(0)}{u^n} - \cdots \cdots - \frac{f^{(n-1)}(0)}{u}$$

2.1 Application of Sumudu transform to some physical problem

Application-I Consider now a particle having mass of 2 grams moves on the x-axis and is attracted a certain point with a force numerically equal to $8x$. If it is initially at rest at O towards origin = 10, find its position at any subsequent time assuming

$a)$ No other force act on it.

$b)$ A damping force act numerically equal to 8 times the instantaneous velocity. These problems can be solved by using Sumudu transform

Solution: From Newton's law, the equation of motion of the particle is
with the initial conditions $X(0) = 10$ and $X'(0) = 0$ taking the Sumudu transform of both sides of (2), we have

$$\frac{d^2X}{dt^2} + 4X = 0 \qquad 0\text{r} \qquad 2\frac{d^2X}{dt^2} = -8X \tag{1}$$

taking Sumudu transform of both sides of (1), we have

$$S\left(\frac{d^2X}{dt^2}\right) + 4\,S(X) = 0$$

using Sumudu transform for derivatives, then

$$\frac{G(u)}{u^2} - \frac{X(0)}{u^2} - \frac{X'(0)}{u} + 4\,G(u) = 0$$

applying initial condition $X(0) = 10$ and $X'(0) = 0$

$$G(u)\left(\frac{1}{u^2} + 4\right) = \frac{10}{u^2}$$

$$G(u) = \frac{10}{u^2}\left(\frac{u^2}{1+4u^2}\right) = \left(\frac{10}{1+4u^2}\right) \ldots \tag{2}$$

taking inverse Sumudu transform of both sides of (2), we have

$$X(t) = 10\,cos2t$$

(b) In this case the equation of motion of the particle is

$$\frac{d^2X}{dt^2} + 4\frac{dX}{dt} + 4X = 0 \ \ldots \tag{3}$$

taking Sumudu transform of both sides of (3), we have

$$S\left(\frac{d^2X}{dt^2}\right) + 4\,S\left(\frac{dX}{dt}\right) + 4\,S(X) = 0$$

using Sumudu transform for derivatives, then

$$\frac{G(u)}{u^2} - \frac{X(0)}{u^2} - \frac{X'(0)}{u} + 4\left(\frac{G(u)}{u} - \frac{X(0)}{u}\right) + 4\,G(u) = 0$$

applying initial condition $X(0) = 10$ and $X'(0) = 0$

$$G(u)\left(\frac{1}{u^2} + \frac{4}{u} + 4\right) = \frac{10}{u^2} + \frac{40}{u}$$

$$G(u)\left(\frac{1+4u+4u^2}{u^2}\right) = \frac{10+40u}{u^2}$$

$$G(u) = \left(\frac{10+40u}{u^2}\right)\left(\frac{u^2}{1+4u+4u^2}\right)$$

$$G(u) = \left(\frac{10+40u}{1+4u+4u^2}\right) = \left(\frac{10+20u+20u}{(1+2u)^2}\right)$$

$$G(u) = \frac{10(1+2u)}{(1+2u)^2} + \frac{20u}{(1+2u)^2} \quad \ldots \qquad (4)$$

taking inverse Sumudu transform of both sides of (4), we have

$$X(t) = 10e^{-2t} + 20t\,e^{-2t}$$

Application-II Consider now a particle having of mass m grams moves along the x-axis and under of a force proportional to its instantaneous speed and in a direction opposite to the direction of motion. Assuming that at $t = 0$ the particle is located at $x = a$ and moving to the right with speed V_0.

Solution: The equation of motion of the particle is

$$m\frac{d^2X}{dt^2} = -\mu\frac{dX}{dt} \quad \ldots \qquad (5)$$

taking Sumudu transform of both sides of (5), we have

$$m\,S\left(\frac{d^2X}{dt^2}\right) + \mu\,S\left(\frac{dX}{dt}\right) = 0$$

using Sumudu transform for derivatives, then

$$\frac{G(u)}{u^2} - \frac{X(0)}{u^2} - \frac{X'(0)}{u} + \frac{\mu}{m}\left(\frac{G(u)}{u} - \frac{X(0)}{u}\right) = 0$$

applying initial condition $X(0) = a$ and $X'(0) = V_0$

$$G(u)\left(\frac{1}{u^2} + \frac{\mu}{mu}\right) = \frac{a}{u^2} + \frac{\mu a}{mu} + \frac{V_0}{u}$$

$$G(u)\left(\frac{m+\mu u}{mu^2}\right) = \left(\frac{am+\mu au + V_0 um}{mu^2}\right)$$

$$G(u) = \left(\frac{am + \mu au + V_0 um}{mu^2}\right)\left(\frac{mu^2}{m + \mu u}\right)$$

$$G(u) = \left[\frac{a(m + \mu u) + V_0 um}{m + \mu u}\right]$$

$$G(u) = \left[a + \frac{V_0 u}{1 + \left(\frac{\mu}{m}\right) u}\right] \quad \dots \qquad (6)$$

taking inverse Sumudu transform of both sides of (6), we have

$$X = X(t) = a - \frac{m}{\mu} V_0 \left[e^{-\frac{\mu}{m}t} - 1\right]$$

$$X = \frac{a\mu + V_0 m}{\mu} - \frac{V_0 m}{\mu} e^{-\frac{\mu}{m}t} \quad \dots \qquad (7)$$

$$\frac{dX}{dt} = V_0 e^{-\frac{\mu}{m}t}$$

when the mass comes to rest, $\frac{dX}{dt} = 0$ hence gives $e^{-\frac{\mu}{m}t} = 0$ and so from (7), $X = \frac{a\mu + V_0 m}{\mu}$

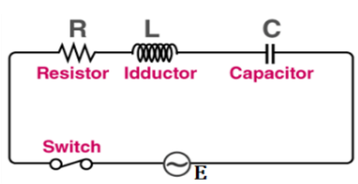
Hence the mass comes to rest at a distance $a + \frac{V_0 m}{\mu}$ from the origin.

2.2 Application of Sumudu Transform to some electrical circuit problem

Application-III The Sumudu Transform can also use to determine the charge (Q) on the capacitors (C) and currents (I) as function of time. Here one applied an alternating electro motive force (emf) $Esin\omega t$.

The differential equation for the determination of the Charge and current in the circuit is gives as[sinR = 0].

Solution:



$$L\frac{dI}{dt} + \frac{Q}{C} = Esin\omega t \quad \dots \qquad (8)$$

$$I = \frac{dQ}{dt} \quad \dots \qquad (9)$$

where $Q(0) = 0, I = \frac{dQ}{dt} = 0$ at $t = 0$ Let $S\{Q(t)\} = Q(u)$

appling Krichhoff's law to the cricuit, we have

$$\mathrm{L}\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{\mathrm{Q}}{\mathrm{C}} = Esin\omega t$$

taking Sumudu transform of both side of (8) and using (9), we have

$$S\left(\mathrm{L}\frac{d^2Q}{dt^2}\right) + S\left(\frac{\mathrm{Q}}{\mathrm{C}}\right) = S(Esin\omega t)$$

$$L\left\{\frac{1}{u^2}Q(u) - \frac{1}{u^2}Q(0) - \frac{1}{u}Q'(0)\right\} + \frac{1}{C}Q(u) = E\left\{\frac{\omega u}{1+\omega^2u^2}\right\}$$

applying the initial conditions $Q(0) = 0, I = Q'(t) = 0$ at $t = 0$

$$L\left\{\frac{1}{u^2}Q(u)\right\} + \frac{1}{C}Q(u) = E\left\{\frac{\omega u}{1+\omega^2u^2}\right\}$$

$$Q(u)\left\{\frac{1}{u^2} + \frac{1}{LC}\right\} = \frac{E}{L}\left\{\frac{\omega u}{1+\omega^2u^2}\right\}$$

$$Q(u)\left\{\frac{1}{u^2} + n^2\right\} = \frac{E}{L}\left\{\frac{\omega u}{1+\omega^2u^2}\right\}, \text{ where } \frac{1}{LC} = n^2$$

$$Q(u)\left\{\frac{1+n^2u^2}{u^2}\right\} = \frac{E}{L}\left\{\frac{\omega u}{1+\omega^2u^2}\right\}$$

$$Q(u) = \frac{E}{L}\left\{\frac{\omega u}{1+\omega^2u^2}\right\}\left\{\frac{u^2}{1+n^2u^2}\right\}$$

by resolving into partial fractions, we have

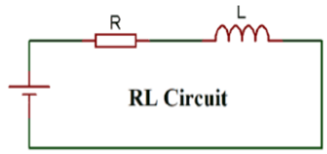
$$Q(u) = \frac{E}{L}\left[\left\{\frac{\omega}{n^2-\omega^2}\right\}\left(\frac{u}{1+\omega^2u^2} - \frac{u}{1+n^2u^2}\right)\right] \dots \qquad (10)$$

taking inverse Sumudu transform of both sides of (10), we have

$$\text{Charge } Q = Q(t) = \frac{E}{L}\left\{\frac{\omega}{n^2-\omega^2}\right\}\left[\left(\frac{\sin(\omega t)}{\omega} - \frac{\sin(nt)}{n}\right)\right]$$

$$\text{Current } I = \frac{dQ}{dt} = \frac{E}{L}\left\{\frac{\omega}{n^2-\omega^2}\right\}(cos\omega t - cosnt)$$

Application-V: Gives $x(0) = 0$, solve
$\boldsymbol{L\frac{dx}{dt} + Rx = Ee^{-at}}$ ... (11)



Solution:

$$\frac{dx}{dt} + \frac{Rx}{L} = \frac{E}{L}e^{-at}, \quad \text{where } x = \frac{dQ}{dt}$$

appling Krichhoff's law to the cricuit, we have

$$\mathrm{L}\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} = Ee^{-at}$$

taking Sumudu transform of both side of (11),

$$S\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right) + \frac{\mathrm{R}}{\mathrm{L}}S(x) = \frac{E}{L}S(e^{-at})$$

$$\frac{1}{u}x(u) - \frac{1}{u}x(0) + \frac{\mathrm{R}}{\mathrm{L}}x(u) = \frac{E}{L}\left(\frac{1}{1+au}\right)$$

applying the initial conditions $x(0) = 0$

$$x(u)\left(\frac{1}{u} + \frac{\mathrm{R}}{\mathrm{L}}\right) = \frac{E}{L}\left(\frac{1}{1+au}\right)$$

$$x(u)\left(\frac{L+uR}{Lu}\right) = \frac{E}{L}\left(\frac{1}{1+au}\right)$$

$$x(u) = \frac{E}{L}\left(\frac{1}{1+au}\right)\left(\frac{u}{1+uR/L}\right)$$

by resolving into partial fractions, we have

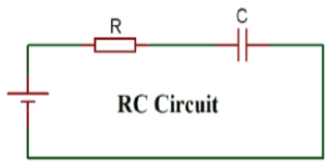
$$x(u) = \frac{E}{L}\left(\frac{1}{a-\frac{R}{L}}\right)\left[\frac{1}{1+\frac{uR}{L}} - \frac{1}{1+au}\right] \quad \ldots \tag{12}$$

taking inverse Sumudu transform of both sides, we have

$$x = x(t) = \frac{E}{L}\left(\frac{L}{R-aL}\right)\left[e^{-at} - e^{-\frac{R}{L}t}\right]$$

Application-IV A resister of R ohms and a capacitor of C Farads are connected in a series with a generator supplying E Volts. At $t = 0$, the charge in the capacitor is zero. Find the charge and current at any time $t > 0$, if $E = E_0 e^{-\alpha t}$

Solution:



appling Krichhoff's law to the cricuit, we have

$$\mathrm{L}\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{\mathrm{Q}}{\mathrm{C}} = E = E_0 e^{-\alpha t}$$

The differential equation $\quad R\frac{dQ}{dt} + \frac{Q}{C} = E_0 e^{-\alpha t}$

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{E_0}{R} e^{-\alpha t} \ldots \tag{13}$$

taking Sumudu transform of both side, we have

$$S\left(\frac{\mathrm{d}Q}{\mathrm{dt}}\right) + \frac{1}{\mathrm{RC}} S(Q) = \frac{E_0}{R} S(e^{-\alpha t})$$

$$\frac{1}{u} Q(u) - \frac{1}{u} Q(0) + \frac{1}{\mathrm{RC}} Q(u) = \frac{E_0}{R}\left(\frac{1}{1 + \alpha u}\right)$$

applying the initial conditions $Q(0) = 0$

$$Q(u)\left(\frac{1}{u} + \frac{1}{\mathrm{RC}}\right) = \frac{E_0}{R}\left(\frac{1}{1 + \alpha u}\right)$$

$$Q(u)\left(\frac{1+bu}{u}\right) = \frac{E_0}{R}\left(\frac{1}{1+\alpha u}\right), \qquad \therefore \frac{1}{\mathrm{RC}} = b$$

$$Q(u) = \frac{E_0}{R}\left(\frac{1}{1 + \alpha u}\right)\left(\frac{u}{1 + bu}\right)$$

by resolving into partial fractions, we have

$$Q(u) = \frac{E_0}{R}\left(\frac{1}{b - \alpha}\right)\left[\frac{1}{1 + \alpha u} - \frac{1}{1 + bu}\right] \ldots \tag{14}$$

taking inverse Sumudu transform of both sides, we have

$$\text{Charge } Q = Q(t) = \frac{E_0}{R}\left(\frac{1}{b-\alpha}\right)\left[e^{-\alpha t} - e^{-bt}\right]$$

$$\text{Current } I = \frac{dQ}{dt} = \frac{E_0}{R}\left(\frac{1}{b-\alpha}\right)\left[-\alpha e^{-\alpha t} + \mathrm{b}e^{-\mathrm{bt}}\right]$$

$$I = \frac{dQ}{dt} = \left(\frac{E_0 C}{1-RC\alpha}\right)\left[\frac{1}{\mathrm{RC}} e^{-\frac{\mathrm{t}}{\mathrm{RC}}} - \alpha e^{-\alpha t}\right], \quad \text{where } \frac{1}{\mathrm{RC}} = b$$

Conclusion: In this paper, we have established the application of Sumudu transform. Sumudu transforms are used to solve application in physical and electrical circuit problem. The result confirms that the Sumudu transform technique is simple and powerful tool. It is anticipated that Sumudu transform method can be used to find analytical solution of linear differential equation. We noted that results of Kamal Transform see in [12] and Sumudu transform are closely connected to each other.

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